Localization of point sources in wave fields from boundary measurements using new sensing principle

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***Abstract*—We address the problem of localizing point sources in 3D from boundary measurements of a wave field. Recently, we proposed the *sensing principle* which allows extracting volumetric samples of the unknown source distribution from the boundary measurements. The extracted samples allow a non-iterative reconstruction algorithm that can recover the parameters of the source distribution projected on a 2-D plane in the continuous domain without any discretization.**

Here we extend the method for the 3-D localization of multiple point sources by combining multiple 2-D planar projections. In particular, we propose a three-step algorithm to retrieve the locations by means of multiplanar application of the sensing principle. First, we find the projections of the locations onto several 2-D planes. Second, we propose a greedy algorithm to pair the solutions in each plane. Third, we retrieve the 3D locations by least squares regression.

***Index Terms*—Sensing principle, finite-rate-of-innovation (FRI), wave equation, source imaging, inverse problem**

I. Introduction

Inverse source problems have wide applications in signal processing and biomedical imaging. Among these, reconstruction of sparse source distributions from boundary measurements have attracted great attention of many researchers recently. In particular, several mathematical models are studied extensively, such as Poisson's equations for identification of current dipolar sources in electroencephalography (EEG) [1], the steady-state diffusion equation for the determination of a light source function in bioluminescence tomography (BLT) [2] and the wave equation for the recovery of heat absorption profile in photoacoustic tomograpghy (PAT) [3]–[5].

Many advanced techniques for the recovery of source distributions aim at super-resolution by exploiting sparsity properties of the underlying source distribution. For example, the low-dimensional signal subspace plays a key role for the MUSIC-type of algorithms to estimate the location of the absorbing regions [6]. Moreover, compressive sensing approaches have been studied recently for radar imaging applications [7].

Here, we focus on the inverse source problem for the wave equation from the boundary measurements of the field. The problem is ill-posed and thus challenging, and we exploit an explicit sparsity prior on the source model (i.e., a collection of point sources) that makes the problem well-posed [8]. Recently, we proposed the sensing principle that allows extracting volumetric samples of the source distribution with a set of well chosen sensing functions [9], [10]. These samples are then used in a non-iterative FRI-like framework [11] to retrieve the projected positions of the source distribution onto a 2-D plane. The key component of the method is the selection of the sensing functions which are used to extract the samples of the source function through surface integration. We have shown before that the localization of the selected families of sensing functions plays a key role in the accuracy of the estimation [9], [10]. Here we propose a multiplanar application of the sensing principle using a well-localized sensing functions for different projections planes. In particular, we propose a three-step algorithm to retrieve the locations of the pointwise source distribution. First, we extract the projected positions onto several 2-D planes. Second, we propose a greedy approach to pair the solutions between projection planes. Third, we reconstruct the 3-D locations from the 2-D paired solutions by a least squares regression.

The paper is organized as follows. In Section 2, we introduce the setting of the problem. In Section 3, we provide the key components of the sensing principle. In Section 4, we develop the proposed method for a 3-D measurement setup. The feasibility of the proposed method is demonstrated with numerical experiments in Section 5.

II. Problem Formulation

Consider an acoustic source distribution inside a volume Ω. In an acoustically homogeneous medium, the inhomogeneous wave equation is described by

$$\nabla^2 p(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 p(\mathbf{r},t)}{\partial t^2} = -H(\mathbf{r},t), \tag{1}$$

where $H(\mathbf{r},t)$ is a general representation of a spatiotemporal source distribution which we further decompose as the product $H(\mathbf{r},t) = A(\mathbf{r})I(t)$, where $A(\mathbf{r})$ is the spatial part and $I(t)$ is the temporal part of the source. In particular, we assume that the temporal behaviour is usually foreknown and we focus on the spatial part of the source function that we characterise as a pointwise source distribution;

$$A(\mathbf{r}) = \sum_{m=1}^{M} c_m \delta(\mathbf{r} - \mathbf{r_m}), \tag{2}$$

where $c_m \in \mathbb{R}$ is the intensity, and $\mathbf{r_m} \in \Omega$ is the location of M point source. With this parametrization the source distribution is completely described by the positions and intensities of M sources with $4M$ parameters. Hence, the goal of the inverse problem is to reconstruct the point sources from the measurements of the wave field $p(\mathbf{r}, t)$ on the surface of the volume, $\partial\Omega$.

III. Sensing Principle

Let us consider the time harmonic solutions of (1)

$$\nabla^2 P(\mathbf{r}, \omega) + \frac{\omega^2}{c^2} P(\mathbf{r}, \omega) = -I(\omega) A(\mathbf{r}), \tag{3}$$

which is the inhomogeneous Helmholtz equation. Without loss of generality, we now consider a specific frequency ω. Based on the second Green's identity, we propose the sensing principle that provides a link between the source function and the measurements such that

$$\langle \Psi, A \rangle = \tag{4}$$
$$\frac{1}{I(\omega)} \oiint_{\partial\Omega} [P(\mathbf{r}, \omega)\nabla\Psi(\mathbf{r}, \omega) - \Psi(\mathbf{r}, \omega)\nabla P(\mathbf{r}, \omega)] \cdot \mathbf{e}_{\partial\Omega} dS,$$

where $I(\omega)$ is a constant that we use to compensate the surface integral and $\Psi(\mathbf{r}, \omega)$ is a sensing function satisfying the homogeneous Helmholz equation in the the volume

$$\nabla^2 \Psi(\mathbf{r}, \omega) + \frac{\omega^2}{c^2} \Psi(\mathbf{r}, \omega) = 0 \text{ in } \Omega. \tag{5}$$

This way the sensing principle allows to extract volumetric samples of the source distribution through a surface integral of the sensor measurements of the acoustic field. Finally, we use the extracted samples by the sensing principle, i.e., $\langle \Psi, A \rangle$ the so-called *generalised samples* to retrieve the parameters of the source function.

IV. Algorithm

We propose a three-step algorithm to estimate the 3-D location of the point sources from the observed acoustic field by means of applying the sensing principle.

A. Planar Projection

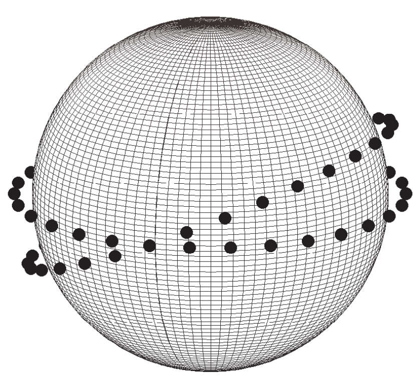

Fig. 1: Poles of the sensing functions in the horizontal XY-plane and after rotation in the X′Y′-plane.

In the first step, we choose a set of sensing functions Ψ satisfying (5) in a general X′Y′Z′coordinate system:

$$\Psi_n(\mathbf{R}\mathbf{r}, \omega) = \frac{e^{j\omega z'/c}}{x' + jy' - a_n}, \ a_n \notin \Omega, \tag{6}$$

where a_n's are the poles of the sensing function on X′Y′-plane located at equidistant angles $a_n = ae^{jn\theta}, n \in [\![0, N-1]\!]$, $|a|$ is greater than the radius of Ω excluding the volume and θ is an arbitrary angle. The matrix $\mathbf{R}$ represents rotation matrix of the coordinate system along the X-axis in a standard right-handed cartesian coordinate system given by

$$\underbrace{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}}_{\mathbf{r}'} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}}_{\mathbf{R}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{r}}. \tag{7}$$

In Fig. 1, we provide a visualisation for the rotation of the poles of the sensing functions on the X′Y′-plane.

Then, we define a polynomial, $Q(X)$ whose roots are the positions of the point sources on the X′Y′-plane:

$$Q(X) = \prod_{m=1}^{M} (X - s'_m) = \sum_{k=0}^{M} q_k X^k \text{ where } s'_m = x'_m + iy'_m \tag{8}$$

where $q_M = 1$. With this selection, the extracted samples of the source function (4) turns into an annihilable equation as follows:

$$\begin{aligned} \langle \Psi_n, A \rangle = \mu_n &= \sum_{m=1}^{M} \frac{c_m e^{i\omega z'_m/c}}{x'_m + iy'_m - a_n} \\ &= \frac{\sum_{m=0}^{M-1} c'_m e^{imn\theta}}{\prod_{m=1}^{M} (x'_m + iy'_m - a_n)} \\ &= \frac{\sum_{m=0}^{M-1} c'_m e^{imn\theta}}{Q(a_n)}, \end{aligned} \tag{9}$$

where c'_m are complex-valued coefficients that do not depend on n nor θ. The sequence $\mu_n \cdot Q(a_n)$, for $n \in [\![0, N-1]\!]$ for some $N \geq 2M + 1$ (i.e., innovation rate given by the FRI sampling [11]), can be annihilated by a known FIR digital filter $h = \{h_k\}$ for $k \in [\![0, N-1]\!]$ characterized as

$$H(z) = \sum_{k \in \mathcal{Z}} h_k z^{-k} = \prod_{k=0}^{M-1} (1 - e^{ik\theta} z^{-1}),$$

where the zeros of the filter are chosen as the poles of (6) on the plane, i.e., $e^{ik\theta}$ for $k \in [\![0, M-1]\!]$. Finally, solving this annihilation system for the coefficients of the polynomial $Q(X)$, the point sources' positions on the X′Y′-plane are found to be the roots of the polynomial Q.

B. Pairing of the Projections

In the second step, we first define an inclusion map so that we can treat the projected points as in $\mathbb{R}^3$. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be an inclusion map defined as

$$f\left(\mathbf{r}'_\mathbf{i} = \begin{bmatrix} x'_i \\ y'_i \end{bmatrix}\right) := \begin{bmatrix} x'_i \\ y'_i \cos(\alpha_i) \\ y'_i \sin(\alpha_i) \end{bmatrix} \tag{10}$$

for each projection point $\mathbf{r_i}'$ on a plane defined by the normal $\mathbf{n_i} = \mathbf{R_i^T n_0}$ (See Fig.2) where $\mathbf{R_i}$ is the rotation matrix of the coordinate system for α_i (7) and $\mathbf{n_0} = [0, 0, 1]^T$ is the normal vector of the standard XY-plane. We propose a naive

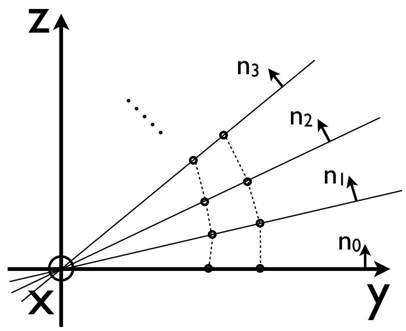

Fig. 2: Visualization of the closest pair of points algorithm for $M = 2$ points projected onto different planes P_i with the normals of the planes $\mathbf{n_i}$

solution for the closest pair problem for two separated sets of points between consecutive projection planes. Indeed, the main idea is to compute the Euclidean distance between all the pairs of points in two sets and then pick the pair with the smallest distance. Let us consider we have P projection planes defined by P_i $i \in [\![0, P-1]\!]$ where each plane has M points to be paired. We assume an initial labelling for the points in plane P_0 with 1 to M. Then, to find the closest pair of points $p \in P_i$ and $q \in P_{i-1}$, we compute the distances between all the $M \times M$ pairs of points and we pick and label the pair with the smallest distance and exclude it from the set. We repeat the same approach for the remaining points. We provide a summary of the method in Algorithm 1 and a visualisation in Fig 2. The method is computed in $O(n^2)$ but can be solved it in $O(n \log n)$ using the recursive divide and conquer approach [13].

Algorithm 1: Closest Pair of Points

Data: $p \in P_i$, for $i \in [[0, P-1]]$
Result: l_p: Labels of $p \in P_i$
begin
  Initialize: Label l_0: 1 to M
  for *i=1* **to** *P-1* **do**
  $P_i^* = P_i$
  while P_i^* *is not empty* **do**
    $p^* = \underset{p \in P_i^*}{\text{argmin}} \min_{q \in P_{i-1}} ||f(p) - f(q)||^2$
    $P_i^* = P_i^* \backslash \{p^*\}$
  Label l_i : Match the labels of p^* and q

C. 3-D Reconstruction of the Positions

In the third step, we solve for the following least squares problem

$$\hat{\mathbf{r}}_m = \underset{\mathbf{r_m}}{\text{argmin}} \sum_{i=0}^{P-1} ||D_i||^2, \ \forall \text{m} \in [\![1, M]\!] \tag{11}$$

where $||D_i||$ is the distance of the solution r_m to the line that passes through the point $f(r_i')$ and parallel to n_i (See Fig. 3):

$$||D_i|| = \frac{||(\mathbf{r_m} - f(\mathbf{r_i'})) \times \mathbf{n_i}||}{||\mathbf{n_i}||},$$

where $\times$ represents the cross product of the two vectors and $||\mathbf{n_i}|| = 1$. In Fig. 3, we provide a visualisation of the solution.

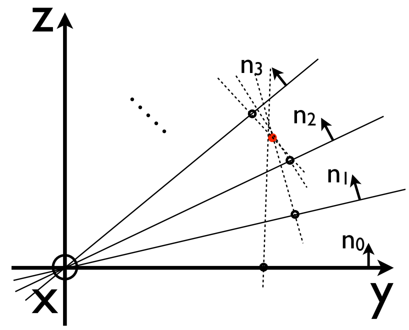

Fig. 3: Visualization of the 3-D reconstruction by least square regression of the distance between the true point (red) and the lines (dashed) defined by the projection points and the normals

V. Experimental Results

We performed numerical experiments to validate our reconstruction algorithm. Specifically, we considered a spherical detection geometry having a radius of 8 cm that is typical for the imaging of breast tissue in a PAT setting using a temporal illumination profile given as $I(t) = \partial/\partial t(e^{-t^2/2\sigma^2})/\sqrt{2\pi\sigma^2}$. The speed of sound is taken as constant $c = 1500$m/s and we assumed that there are 134^2 sensors uniformly positioned on the surface. We focused on the localisation accuracy of our method.

We define the reconstruction error per point source by

$$\text{RMSE} = \sqrt{\frac{1}{M} \sum_{i=1}^{M} ||\mathbf{r}_m - \hat{\mathbf{r}}_m||^2}$$

where $\mathbf{r}_m$ is the true position, $\hat{\mathbf{r}}_m$ is the estimated position. In Fig. 4, we compare the reconstruction accuracy using the frequency samples at 200 KHz of the sensor data at 20 dB for varying number of projections such that the angle between the planes is $\frac{\pi}{2P}$. In Fig. 4, we demonstrate the improvement obtained by increased number of projections in which we achieve about mm reconstruction accuracy among a radius

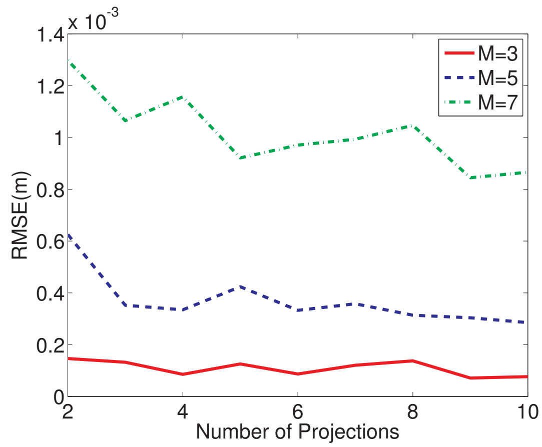

Fig. 4: RMSE results of an average of 50 independent realisations for varying number of projections for $M = 3, 5$ and 7 using the frequency samples at 200 KHz of the sensor data at 20 dB

of 8cm. We conclude that multiplanar approach performs accurate localization once the sensing principle is applied on sufficient number of projection planes, i.e., small projection angle between the planes.

VI. Conclusions

In sum, we proposed a non-iterative algorithm for the detection of point absorbers in three dimensional wave equation from the boundary measurements. The key component of the method is the selection of the sensing function that is used to extract the generalized samples by the surface integral. Here, we demonstrate that a well localised family of sensing functions with the proposed framework to build the solution in 3D from 2D projections can achieve accurate results even for the low SNR regime.

For simplicity of the discussion, we provide the method that combines the projected solutions using a simple rotation of the coordinate system along the X-axis only. However, a general rotation in three dimension can be obtained from three basic rotation matrices along X,Y, and Z-axes. Therefore, the idea can be easily generalized to a framework that combines the projections from any rotation as a composition of the rotations along the three axes.

Sparse models for the inverse source problems from overdetermined boundary field measurements remain as a promising research area of further research. The current work focuses on the systems governed by the wave equation, however the framework can be applied to similar problems encountered in different domains. Moreover, we also consider possibility and feasibility of the proposed method in real applications.

VII. Acknowledgments

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