Preprint submitted to
SAMPLING THEORY IN SIGNAL AND IMAGE PROCESSING

DVI file produced on
September 21, 2014

Finite-rate-of-innovation for the inverse source problem of radiating fields

Zafer Doğan
Medical Image Processing Lab, Institute of Bioengineering, School of Engineering, EPFL†
Medical Image Processing Lab, Department of Radiology, University of Geneva‡
†CH-1015 Lausanne, Switzerland, ‡CH-1211 Geneva, Switzerland
zafer.dogan@epfl.ch

Thierry Blu
Department of Electronic Engineering, Chinese University of Hong Kong
Shatin N.T. Hong Kong
thierry.blu@m4x.org

Dimitri Van De Ville
Medical Image Processing Lab, Institute of Bioengineering, School of Engineering, EPFL†
Medical Image Processing Lab, Department of Radiology, University of Geneva‡
†CH-1015 Lausanne, Switzerland, ‡CH-1211 Geneva, Switzerland
dimitri.vandeville@epfl.ch

Abstract

Finite-rate-of-innovation (FRI) is a framework that has been developed for the sampling and reconstruction of specific classes of signals, in particular non-bandlimited signals that are characterized by finitely many parameters. It has been shown that by using specific sampling kernels that reproduce polynomials or exponentials (i.e., satisfy Strang-Fix condition), it is possible to design non-iterative and fast reconstruction algorithms. In fact, the innovative part of the signal can be reconstructed perfectly using Prony's method (the annihilating filter).

In this paper, we propose an adapted FRI framework to deal with the inverse source problem of radiating fields from boundary measurements. In particular, we consider the case where the source signals are modelled as stream of Diracs in 3-D and, we assume that the induced field governed by the Helmholtz equation is measured on a boundary. First, we propose a technique, termed "sensing principle –also known as the reciprocity gap principle– to provide a link between the physical measurements and the source signal through a surface integral. We have shown that it is possible to design sensing schemes in complex domain using holomorphic functions such that they allow to determine the positions of the sources with a non-iterative algorithm using an adapted annihilating filter method.

Key words and phrases : Finite-rate-of-innovation, sensing principle, Helmholtz equation, wave equation, inverse source problem, annihilating filter method

2000 AMS Mathematics Subject Classification — 15A29, 31A25, 35J05

1 Introduction

The inverse source problem (ISP) is of interest and importance across many branches of physics, mathematics, engineering and medical imaging. Among these, reconstruction of source distributions from boundary measurements of radiating fields have attracted great attention of many researchers. In particular, the Helmholtz equation that is the fundamental model for the radiation and wave propagation has been studied extensively for various electromagnetic and scalar fields [14, 9]. In general, the underlying physical system assumes a well-posed forward model but, they usually suffer from having an ill-posed inverse problem in terms of uniqueness, stability, and existence of a solution. Typically, one needs additional assumptions about the source distribution to force uniqueness of the solution by either imposing smoothness properties of the distribution or by assuming a parametric source model.

The standard solutions of the ISP rely on iteratively fitting of a source model using the forward model. In this case, the sparsity assumption of the source signal plays a key role to regularize the solution with an optimisation framework [12]. Recently, compressive sensing approaches have been employed [5] for the detection of sparse objects from the field measurements.

There exists several approaches for the ISP that assume parametric source models. In particular, the mathematical uniqueness and local stability of the source distributions modelled as point sources have been proven [4]. Moreover, there are several other parameter estimation frameworks in which the computational burden of forward model fitting can be dealt with efficient algorithms [13]. For example, the method known as "reciprocity gap" concept [2] which is essentially an application of Green's theorem has been recently applied ISP from boundary measurements of a Poisson's field [4]. The method transforms a scalar product between the source signal and and a test function to a boundary integral of the measurements and the test function [4].

In the signal processing world, several sampling and reconstruction methods have recently been proposed for specific classes of signals [27]. The common feature of these signals is that they have a parametric representation with finite number of parameters and are, therefore, called the signals with finite-rate-of-innovation (FRI) [11]. It was shown that it is possible to reconstruct the parameters of streams of Diracs, piecewise polynomials and piecewise sinusoidal by using adequate sampling kernels that are able to reproduce polynomials or exponentials. The results of FRI sampling has been extended for multidimensional signals [22] and recently for arbitrary sampling kernels [25]. Moreover, there has been several studies that develops similar parametric estimation frameworks for different applications such as sampling of pulse streams in ultrasound tomography [24] and spike detection from calcium imaging [18].

Recently, the theory of FRI has been applied to the problem of detecting parametric point sources from boundary measurements of a field generated by the Poisson's equation [15]. The method proposes analytic sensing functions to map back the boundary measurements to underlying generator signal and shows that it is possible to develop non-iterative reconstruction algorithms to retrieve the innovation parameters. The results has been also extended to a multi–layer head model to detect the epileptic foci in electroencephalograhy (EEG) data [16].

In this work, we focus on the ISP from boundary measurements of radiating fields governed by the Helmholtz equation. A typical measurement setup is shown in Figure 1. We exploit an explicit sparsity prior on the source model being a 3-D stream of Diracs such that the only innovation parameters are the locations and weights. Then, we develop a new framework that allows us to identify parametric source models from boundary measurements of a radiating field. For that, we extend our recent work [10] to general sensing functions derived from holomorphic functions in the complex plane. Our contributions are two fold:

- We apply the reciprocity gap concept to the sensing functions which are solutions to homogeneous Helmholtz equation and show that these functions can be used to extract the innovation parameters of the source signal;
- We propose families of sensing functions that are holomorphic functions that generate complex polynomials of N-th degree and we show that these sensing functions have a spatial localization that can be controlled so that they allow to sense the influence of nearby pointwise sources.

This approach brings together several attractive features: (1) the 2-D projections of the locations onto several planes are decoupled; (2) several 2-D projections are combined to retrieve the 3-D locations with a tomographic approach; (3) the solution to the forward model is not necessary; (4) the method is locally adaptive thanks to the generalization of the holomorphic functions to reproduce N-th order polynomials.

1.1 Forward problem

The forward problem considers the radiation of waves from a real-valued spatio-temporal source distribution $q(r,t)$ embedded in an infinite, homogeneous medium. The real-valued radiating wave field satisfies the inhomogeneous scalar wave equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] u(\mathbf{r},t) = q(\mathbf{r},t), \tag{1}$$

for all the space and time, where c is the speed of wave propagation in the medium. The source term $q(\mathbf{r},t)$ is assumed to be compactly supported in the space-time region $S_0|\mathbf{r} \in \Omega_0, t \in [0,T_0]$, where Ω is the spatial volume and $[0,T_0]$

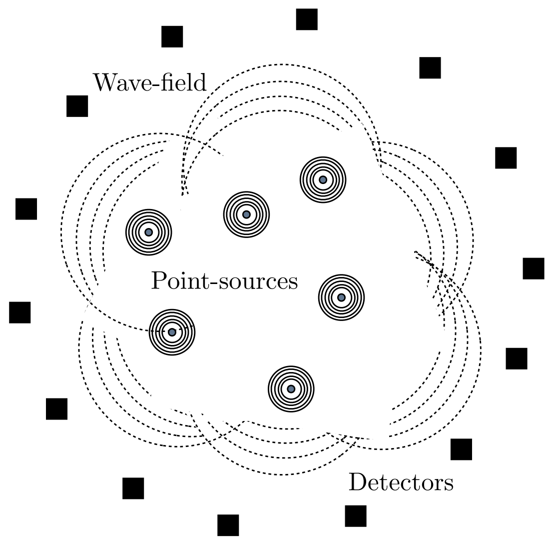


Figure 1: Typical measurement setup and the simplified view of the radiating wavefronts

is the interval of time over which the source is present. The solution to (1) is not unique. In particular, adding any solution of the homogeneous wave equation to $u(\mathbf{r}, t)$ will be again a solution to (1). Therefore, it is necessary to specify initial conditions in the form of Cauchy conditions.

Primarily, the harmonic solution of the wave equation is of more interest in various application. This is mainly because the wave equation applies only to non-dispersive and non-attenuating medium, whereas the counterpart Helmholtz equation describes the radiation of waves in a general dispersive medium

$$\left[\nabla^2 + k^2)\right] U(\mathbf{r}, \omega) = Q(\mathbf{r}, \omega), \tag{2}$$

where $k^2(\omega) = \omega^2/c^2(\mathbf{r})$ is the wavenumber and $c(\mathbf{r})$ is speed of wave propagation in an inhomogenous medium. Hence, the Helmholtz equation is considered to be the fundamental governing equation of radiation and wave propagation. As was the case for (1), (2) does not posseses a unique solution and, in particular one needs to determine the boundary conditions that are dictated by the physics of the problem on the measurement. Then, if the source term, $Q(\mathbf{r}, \omega)$ is known, the solution to (1) can be written as

$$U(\mathbf{r}, \omega) = \int_{\Omega} d^3r' G_+(\mathbf{r} - \mathbf{r}', \omega) Q(\mathbf{r}', \omega), \tag{3}$$

where $G_+(\mathbf{r})$ is the retarded Green's function of the Helmholtz equation defined as the solution to the partial differential equation

$$\left[\nabla^2 + k^2\right] G_+(\mathbf{r} - \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}'). \tag{4}$$

In the case of the radiating fields in free-space, the physically appropriate boundary condition is that the Green's function satisfies the Sommerfeld radiation condition which is equivalent to the requirement of causality in time domain [9]. Hence, the resulting Green's function is known as the retarded Green's function

$$G_+(\mathbf{r}-\mathbf{r}',\omega) = -\frac{1}{4\pi}\frac{e^{ik\|\mathbf{r}-\mathbf{r}'\|}}{\|\mathbf{r}-\mathbf{r}'\|} \tag{5}$$

that represents an outgoing-wave.

1.2 Inverse source problem

The inverse source problem (ISP) considers finding the source term $Q(\mathbf{r},\omega)$ from the knowledge of the radiating field $U(\mathbf{r},\omega)$. The solution to the ISP is trivial, if the field $U(\mathbf{r},\omega)$ is known over S_0. Indeed, one can simply apply the D'Alembertian operator $\mathcal{W} = \nabla^2 + k^2$ to the field to recover the source term according to the Helmholtz equation. However, in practical situations, the field can only be measured in restricted region that lie outside the source's space-time support S_0. In particular, the field $U(\mathbf{r},\omega)$ and its normal derivative $\partial U(\mathbf{r},\omega)/\partial n'$ are available on a closed surface $\partial\Omega$.

The standard solution of the source term by the boundary data is given by the Porter-Bojarski (PB) integral equation,

$$Q(\mathbf{r},\omega) = -\int_{\partial\Omega} dS' \left[U(\mathbf{r},\omega)\frac{\partial G_-(\mathbf{r}-\mathbf{r}',\omega)}{\partial n'} - G_-(\mathbf{r}-\mathbf{r}',\omega)\frac{\partial U(\mathbf{r},\omega)}{\partial n'}\right], \mathbf{r}\in\Omega \tag{6}$$

where $G_- = G_+^*$ is known as the retarded Green's function representing an incoming-wave, hence the solution is referred as the back-propagated-field solution [9].

The classical treatment of the problem based on (6) has several limitations that include being limited to non-dispersive media and the requirement of having full data set over a closed surface surrounding the source. As an alternative, the problem can be cast in a Hilbert space formulation

$$TQ = f, \tag{7}$$

where $T : \mathcal{H}_Q \to \mathcal{H}_f$ is a linear mapping from a Hilbert space of source functions $\mathcal{H}_Q$ to Hilbert space of measurements $\mathcal{H}_f$, f is the data and Q is the source terms. With this formulation, the ISP will also apply to the cases of incomplete data as well as to dispersive medium. However, as the two Hilbert spaces $\mathcal{H}_Q$ and $\mathcal{H}_f$ are generally different and the linear operator T is not generally Hermitian, inverting such mappings for the source in terms of data would generally require computationally heavy finite element techniques [9].

2 Finite rate of innovation for the Helmholtz equation from boundary measurements

2.1 FRI in a nutshell

In standard FRI, the sampling kernels need to satisfy the Strang-Fix condition on a uniform grid so that the sampling scheme will allow to reproduce some polynomials or exponentials. In this case, the uniformly sampled FRI signal with the proper sampling kernel allows to extract the moments or Fourier measurements of the unknown signal which will be further used in the reconstruction scheme [7, 11, 27].

Now, assume that we want to retrieve an input signal $x(t)$ of stream of K Diracs $x(t) = \sum_{k=0}^{K-1} a_k \delta(t - t_k)$, where $a_k \in \mathbb{R}$ are the amplitudes and $t_k \in \mathbb{R}$ are the time locations of the Diracs. We assume that the signal is sampled uniformly after filtering with a kernel $\varphi(t)$ and obtain the samples $y_n = \left\langle x(t), \varphi\left(\frac{t}{T} - n\right)\right\rangle$, where $n = 0, \ldots, N-1$. Moreover, we assume that $\varphi(t)$ is an exponential reproducing kernel of compact support satisfying

$$\sum_{n \in \mathbb{Z}} c_{m,n} \varphi(t - n) = e^{\alpha_m t}, \tag{8}$$

for proper coefficients $c_{m,n}$ with $m = 0, \ldots, P$ and $\alpha_m = \alpha_0 + m\lambda \in \mathbb{C}$ for $m = 0, \ldots, P$. Then, FRI theory states that linearly combing the samples y_n with the coefficients $c_{m,n}$, one achieves a power sum series that can be annihilated by an FIR filter

$$s_m = \sum_{n=0}^{N-1} c_{m,n} y_n = \sum_{k=0}^{K-1} x_k u_k^m, \tag{9}$$

where $x_k = a_k e^{\alpha_0 t_k / T}$ and $u_k = e^{\lambda t_k / T}$. Defining a filter with z-transform $h(z) = \sum_{m=0}^{K} h_m z^{-m} = \prod_{k=0}^{K-1}(1 - u_k z^{-1})$, that is, its roots correspond to the values u_k to be found. Then, it follows that $h_m * s_m = 0$. Hence, the zeros of the filter uniquely define the values u_k provided that t_k's are distinct.

2.2 Innovation signal for radiating field

We consider a 4-D signal model with M points sources given inside a region $\Omega \subset \mathbb{R}^3$, enclosed by a surface $\partial\Omega$ where the measurements of the field are taken. Assume that the spatial distribution of the sources is given by a set of points at locations $\{\mathbf{r}_m\}_{m=1}^{M} \in \Omega$. The mth source's waveform is given by the signals temporal Fourier transform $s_m(\omega)$. These signals may represent any type of acoustic source such as music, speech, or noise. Hence, the total source distribution inside Ω is then described by

$$Q(\mathbf{r}, \omega) = \sum_{m=1}^{M} s_m(\omega) \delta(\mathbf{r} - \mathbf{r}_m) \tag{10}$$

where the only free parameters in the signal $Q(\mathbf{r}, \omega)$ are the locations $\mathbf{r_m}$ and the Fourier coefficients of the m^{th} source signal for a given frequency ω. The generated wave field according to (1) is observed with some detectors located $\{\mathbf{r}_d\}_{d=1}^{D} \in \partial\Omega$ known by the measurement setup. Finally, the problem can be stated as follows.

Problem 2.1. *Given a set of measurements of a propagating wave field $U(\mathbf{r}, \omega)$ and its normal derivative $\partial U(\mathbf{r}, \omega)/\partial n'$ for a set of points $\mathbf{r}_d$ on $\partial\Omega$, find the source locations $\{\mathbf{r}_m\}_{m=1}^{M} \in \Omega$ satisfying* (2)*.*

2.3 Sensing Kernels

We start by defining the *Sensing Principle* as the current work differentiates from the classical FRI–sampling problems.

Definition 2.2 (Sensing Function)**.** *Let Ψ be a function that satisfy*

$$\nabla^2\Psi(\mathbf{r}, \omega) + \frac{\omega^2}{c^2}\Psi(\mathbf{r}, \omega) = 0 \text{ in } \Omega, \tag{11}$$

then we coin the term ***sensing function*** *for Ψ by noting that this set is a subset of the the space of solutions to homogeneous Helmholtz equation.*

Proposition 2.3. *Assuming the field and the normal derivative of the wave field are available on the boundary $\partial\Omega$, and one chooses Ψ satisfying* (11)*, then one can "sense" the source signal through the surface integral:*

$$\langle \Psi, Q \rangle = \oint_{\partial\Omega} \left[\Psi(\mathbf{r}, \omega)\frac{\partial}{\partial n}U(\mathbf{r}, \omega) - U(\mathbf{r}, \omega)\frac{\partial}{\partial n}\Psi(\mathbf{r}, \omega) \right] dS, \tag{12}$$

where the partial derivatives $\frac{\partial}{\partial n}$ are directed outward (from the interior to exterior) and we call $\langle \Psi, Q \rangle$ the ***generalized samples*** *to differentiate from the field measurements.*

Proof. Let $\Psi(\mathbf{r}, \omega)$ and $U(\mathbf{r}, \omega)$ be any two complex functions of position, and $\partial\Omega$ be a closed surface surrounding a volume Ω. If $\Psi(\mathbf{r}, \omega)$, $U(\mathbf{r}, \omega)$, and their first and second partial derivatives are well-defined within Ω and on $\partial\Omega$, respectively, then the second Green's identity states that

$$\int_{\Omega} (U\nabla^2\Psi - \Psi\nabla^2 U) dV = \oint_{\partial\Omega} \left(\Psi\frac{\partial}{\partial n}U - U\frac{\partial}{\partial n}\Psi \right) ds. \tag{13}$$

When the sensing function Ψ is chosen from (11), we obtain

$$\oint_{\partial\Omega} \left(\Psi\frac{\partial}{\partial n}U - U\frac{\partial}{\partial n}\Psi \right) \cdot ds = \int_{\Omega} \Psi(\mathbf{r}, \omega)Q(\mathbf{r}, \omega)d\mathbf{r} \tag{14}$$

$$= \langle \Psi, Q \rangle . \tag{15}$$

Hence, the sensing principle that follows from the second Green's identity allows extracting generalized samples of the source term with the sensing function, which creates a link between the model parameters and the measurements on the surface. □

Many functions that satisfy the Strang–Fix conditions can be extended to multidimensional space by the tensor product [22]. For example, symmetric B-spline, biorthogonal B-spline and orthogonal Daubechies scaling functions [8]. However, the condition that is given by (11) prevents straight forward extension of the FRI–theory to multidimensional sensing problems. Moreover, there has been various attempts to define functions that satisfy generalized Strang–Fix condition for scattered data quasi-interpolation. For conditionally positive definite functions such as multiquadratics, thin-plate splines and polyharmonic B-splines, it can be checked that Strang–Fix conditions are satisfied, by taking finite linear combinations of shifted functions [26]. However, all conditionally positive definite radial basis functions are unbounded and not compactly supported [28]. Therefore, eigenfunctions of the Laplacian, which are solutions of (11), cannot be used to retrieve parameters through reproducing polynomials or exponentials as in the standard FRI. For that reason, we propose various families of sensing functions that splits the problem into pieces in which there exists efficient algorithms to retrieve the innovations of the signal.

Proposition 2.4 (Sensing functions based on 2D harmonic)**.** *Let $\phi(x, y)$ be a solution to $\nabla^2\phi(x, y) = 0$, then any function*

$$\Psi(x, y, z) = e^{\pm ikz}\phi(x, y) \tag{16}$$

would be a solution of the sensing equation given in (11)*.*

Proof. Choosing Ψ as in (16) and developing (11), we have:

$$\begin{aligned}
\left[\nabla^2 + k^2\right]\Psi(x, y, z) &= \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right] e^{\pm ikz}\phi(x, y) \\
&= e^{\pm ikz}\underbrace{\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\phi(x, y)}_{=0} + \left[(\pm ik)^2 + k^2\right] e^{\pm ikz}\phi(x, y) \\
&= 0.
\end{aligned}$$

The term that appears to be zero in the second line is by the definition of a harmonic function and it is worth mentioning that if the harmonic function possesses a singularity, it has to be outside of the domain Ω to satisfy (11) and to be a valid sensing function. □

Now, we consider a 2-D sensing setup with the proposed 3-D sensing functions and we ask the question whether the harmonic part of the sensing function can also satisfy the Strang-Fix conditions to reproduce some polynomials or exponentials.

(a) *Polynomial reproducing kernels:* Consider any compactly supported kernel given by the tensor product of two 1-D functions $\phi(x)$ and $\phi(y)$ that can reproduce polynomials x^α and y^β, respectively, where $\alpha, \beta \in \{1, \dots, n\}$ and $x, y \in \mathbb{R}$. Assuming unit sampling period along each direction, this means that the kernel $\varphi(x, y)$ satisfies

$$\sum_j \sum_k C_{j,k}^{\alpha,\beta} \phi(x - j, y - k) = x^\alpha y^\beta, \tag{17}$$

where α and β are the degrees of the polynomials that the kernel can reproduce along x- and y-directions.

Proposition 2.5. *There is no kernel that can reproduce real polynomials of degree more than 2 that will satisfy both the sensing principle* (11) *and the Strang-Fix conditions* (17).

Proof. Let $\phi(x, y)$ be the harmonic function; i.e., $\nabla^2 \phi(x, y) = 0$ in Ω. Hence, (17) must satisfy the same equation

$$\begin{aligned} \text{LHS} &= \sum_j \sum_k C_{j,k}^{\alpha,\beta} \nabla^2 \phi(x - j, y - k) = 0 \\ \text{RHS} &= \nabla^2 x^\alpha y^\beta = \alpha(\alpha - 1) x^{\alpha-2} y^\beta + \beta(\beta - 1) x^\alpha y^{\beta-2}. \end{aligned}$$

The RHS is zero only for $\alpha, \beta \in \{1, 2\}$, which yields a contradiction for polynomials of degree $n > 2$. □

(b) *Exponential reproducing kernels:* Consider any compactly supported kernel given by the tensor product of two 1-D functions $\phi(x)$ and $\phi(y)$ that can reproduce exponentials $e^{\alpha_p x}$ and $e^{\beta_q y}$, respectively, where $\alpha_p = \alpha_0 + p\lambda, \beta_q = \beta_0 + q\gamma$ with $p, q \in \{1, \dots, n\}$ and $x, y \in \mathbb{R}$. Assuming unit sampling interval along each direction, this means that the kernel $\phi(x, y)$ satisfies

$$\sum_r \sum_s C_{r,s}^{p,q} \phi(x - r, y - s) = e^{\alpha_p x} e^{\beta_q y}, \tag{18}$$

where α and β are the degrees of the polynomials that the kernel can reproduce along x- and y-directions.

Proposition 2.6. *There exists kernels that can reproduce exponentials that will satisfy both the sensing principle* (11) *and the Strang-Fix conditions* (17) *provided that $\alpha_p = \pm i \beta_q$ for all p, q.*

Proof. Let $\varphi(x, y)$ be the harmonic function; i.e., $\nabla^2\phi(x, y) = 0$ in Ω. Hence, (18) must satisfy the same equation

$$\text{LHS} = \sum_r \sum_s C^{p,q}_{r,s} \nabla^2 \phi(x - r, y - s) = 0$$
$$\text{RHS} = \nabla^2 e^{\alpha_p x} e^{\beta_q y} = (\alpha_p^2 + \beta_n^2) e^{\alpha_p x} e^{\beta_q y} = 0.$$

□

Consider a general harmonic function that has no singularity in Ω and a stream of Diracs with $Q(\mathbf{r}, \omega) = \sum_{m=1}^{M} s_m(\omega)\delta(\mathbf{r} - \mathbf{r}_m)$ for a given frequency ω, $\mathbf{r}^T = [x, y, z]$ and $\mathbf{r_m}^T = [x_m, y_m, z_m]$. We consider the generalized samples on a uniform grid given by

$$M_{r,s} = \left\langle Q(\mathbf{r}, \omega), \Phi\left(\frac{x}{T_x} - r, \frac{y}{T_y} - s, z\right)\right\rangle \tag{19}$$
$$= \iiint_{\mathbb{R}^3} Q(\mathbf{r}, \omega)\phi\left(\frac{x}{T_x} - r, \frac{y}{T_y} - s\right) e^{\pm ikz} dxdydz$$

where $T_x, T_y \in \mathbb{R}^+$ are the sensing intervals along x- and y-directions.

Consider a set of 3-D Dirac distribution $Q(\mathbf{r}, \omega) = \sum_{m=1}^{M} s_m(\omega)\delta(\mathbf{r} - \mathbf{r}_m)$ for a given frequency ω, $\mathbf{r}^T = [x, y, z]$ and $\mathbf{r_m}^T = [x_m, y_m, z_m]$. Here, we provide an algorithm in the classical FRI–fashion and we refer interested reader to [11] for the details of the method.

1) *Retrieve the FRI samples of the signal:* We denote, $\mu_{p,q} = \sum_r \sum_s C^{p,q}_{r,s} M_{r,s}$ the weighted sum of the generalized samples, where the weights are those from (18) that reproduce $e^{\alpha_p x} e^{\beta_q y}$. We have

$$\begin{aligned}
\mu_{p,q} &= \sum_r \sum_s C^{p,q}_{r,s} M_{r,s} \\
&\overset{(a)}{=} \sum_r \sum_s C^{p,q}_{r,s} \left\langle Q(\mathbf{r}, \omega), \Phi\left(\frac{x}{T_x} - r, \frac{y}{T_y} - s, z\right)\right\rangle \\
&\overset{(b)}{=} \left\langle Q(\mathbf{r}, \omega), e^{\pm ikz} \sum_r \sum_s C^{p,q}_{r,s} \phi\left(\frac{x}{T_x} - r, \frac{y}{T_y} - s\right)\right\rangle \\
&\overset{(c)}{=} \left\langle \sum_{m=1}^{M} s_m(\omega)\delta(\mathbf{r} - \mathbf{r}_m), e^{\pm ikz} e^{\alpha_p x} e^{\beta_q y}\right\rangle \\
&\overset{(d)}{=} \sum_{m=1}^{M} s_m(\omega) e^{\pm ikz_m} e^{\alpha_p x_m} e^{\beta_q y_m}
\end{aligned} \tag{20}$$

where (a) follows from the definition of $M_{r,s}$ in (19), (b) from the definition of Ψ in (16); (c) from the definition of $Q(\mathbf{r},\omega)$ and (d) from the exponential reproducing property in (18).

2) *Annihilation along x- and y- axis:* Note that (20) can be written as a power series

$$\mu_{p,\cdot} = \sum_{m=1}^{M} c_m u_m^p \tag{21}$$

with $c_m = s_m(\omega)e^{\pm ikz_m}e^{\beta_q y_m e^{\alpha_0 x_m}}$ and $u_m = e^{\lambda x_m}$ along each row of $\mu_{p,q}$. Here, the choice $\alpha_p = \alpha_0 + p\lambda$ makes the sum as a power series that can be annihilated with an annihilating filter. Hence the sequence $\{x_m\}_{m=1}^M$ can be retrieved from the FRI–samples $\mu_{p,q}$ using the annihilating filter method also known as Prony's method [23]. Let h_p with $p = 0, \dots, P$, be the filter with z-transform $H(z) = \sum_{p=0}^M h_p z^{-p} = \prod_{m=1}^M (1 - u_m z^{-1})$, that is its roots correspond to the values u_m to be found. Then, it follows that h_p annihilates the observed sequence $\mu_{p,\cdot}$:

$$h_p * \mu_{p,\cdot} = \sum_{i=0}^{M} h_i \mu_{p-i,\cdot} = \sum_{m=1}^{M} c_m u_m^p \underbrace{\sum_{i=1}^{M} h_i u_m^{-i}}_{H(u_m)} = 0. \tag{22}$$

Then, the zeros of the this filter uniquely defines the values u_m provided that the x_m's are distinct. Moreover, the same procedure can be followed to retrieve the innovations $\{y_m\}_{m=1}^M$ by defining another annihilating filter for $\beta_q = \beta_0 + q\gamma$.

As a final remark for this section, we note that the origin of the sensing grid and step size T_x and T_y have to be chosen such that there exist no singularity in the volume Ω so that (11) will be satisfied for every sensing point of the grid in case the function ϕ possesses a singularity.

(c) *Holomorphic kernels:* In this part, we propose to work with holomorphic functions that are a subset of harmonic functions rather than the general harmonic functions. We first note that this allows to reduce the dimension of the problem, that is, the pairs $\{x_m, y_m\}_{m=1}^M$ will be represented by complex numbers $\xi_m = x_m + iy_m$. Then, we showed that the exponential reproduction constrain of part (b) can be relaxed with a proper design of sensing positions of the generalized samples. We propose to take these samples at equidistant angles (using polar representation) on the complex domain that will allow to construct an annihilation filter apiori so that the parameters of a characteristic polynomial in which the zeros are defined as the positions on the complex plane can be retrieved with a non-iterative algorithm.

We start by noting that, if a complex-valued function $\varphi(\xi)$ of a single complex variable $\xi = x + iy$ is complex differentiable, (i.e., holomorphic), it is also a harmonic function, i.e., $\nabla^2 \varphi = 0$, [1]. Polynomial functions in ξ with complex coefficients, sine, cosine and the exponential function are some examples of holomorphic functions on $\mathbb{C}$. In this work, we only consider N-th degree polynomials given by

$$\varphi(\xi) = \sum_{l=0}^{N} a_l \xi^l = \prod_{l=1}^{N} (\xi - s_l), \tag{23}$$

where s_l are the zeros of the holomorphic functions that are located in the complex plane in a region S_0 with a radius $s_0 = \max_l |s_l|$. Then, we propose to introduce the zeros of the holomophic function as the poles of the sensing function

$$\Psi(\xi, z) = e^{ikz} \left(\varphi(\xi)\right)^{-1}, \tag{24}$$

such that one can acquire the generalised samples by

$$\mu_n = \langle Q(\mathbf{r}), \Psi(\xi - a_n, z) \rangle, \tag{25}$$

where the sensing positions, (i.e., $a_n = r_n e^{i\alpha_n}$), are not on a uniform grid in the complex plane, but located at equidistant angles satisfying $\alpha_n = \alpha_0 + \lambda n$ with arbitrary α_0, λ and $r_n \geq s_0 + R$ with R being the radius of the volume Ω to ensure that no singularity exists in the volume for different sensing positions.

Proposition 2.7. *Consider a stream of 3-D Diracs*

$$Q(\mathbf{r}, \omega) = \sum_{m=1}^{M} s_m(\omega) \delta(\mathbf{r} - \mathbf{r}_m)$$

for a given frequency ω, $\mathbf{r}^T = [x, y, z]$ and $\mathbf{r_m}^T = [x_m, y_m, z_m]$, the 3-D complex sensing functions in (24) that are designed using the holomorphic functions that introduce N-th order poles allow non-iterative reconstruction algorithm to retrieve the locations of the sequence $\{\xi_m = x_m + iy_m\}_{m=1}^{M}$.

Proof. The sensing samples in (25) will satisfy

$$\mu_n = \left\langle Q(\mathbf{r},\omega), \frac{e^{ikz}}{\varphi(\xi - a_n)} \right\rangle \tag{26}$$

$$\stackrel{(a)}{=} \sum_{m=1}^{M} \frac{s_m(\omega)e^{ikz_m}}{\varphi(\xi_m - a_n)} \tag{27}$$

$$\stackrel{(b)}{=} \frac{\sum_{m=1}^{M} s_m(\omega)e^{ikz_m} \prod_{\substack{i=1\\ i\neq m}}^{M} \varphi(\xi_i - a_n)}{\prod_{m=1}^{M} \varphi(\xi_m - a_n)} \tag{28}$$

$$\stackrel{(c)}{=} \frac{\sum_{m=1}^{M} s_m(\omega)e^{ikz_m} \prod_{\substack{i=1\\ i\neq m}}^{M} \prod_{l=1}^{N}(\xi_i - s_l - a_n)}{\prod_{m=1}^{M} \prod_{l=1}^{N}(\xi_m - s_l - a_n)} \tag{29}$$

$$\stackrel{(d)}{=} \frac{\sum_{m=0}^{(M-1)N} s'_m a_n^m}{P(a_n)} \tag{30}$$

where (a) follows from the linearity of the inner product, (b) from combining each terms in (a), (c) from the definition of the φ in (23), the numerator of (d) follows from the fact that the numerator of (c) can be rewritten as a polynomial with respect to a_n with at most $(M-1)N$ zeros where s'_m are complex valued coefficients that do not depend on a_n and the denominator of (d) follows from defining a characteristic polynomial

$$P(x) = \sum_{m=0}^{MN} p_m x^m = \prod_{i=1}^{M} \prod_{l=1}^{N} (\xi_m - s_l - x) \tag{31}$$

where p_m are the coefficients to be found such that $p_{MN} = 1$. Then, defining a new sequence

$$u_n = \mu_n P(a_n) = \sum_{m=1}^{(M-1)N} s'_m a_n^m = \sum_{m=1}^{(M-1)N} c'_m u_m^n, \tag{32}$$

where $c'_m = s' m r_n e^{i\alpha_0 m}$ and $u_m = e^{i\lambda m}$. Here, the choice $\alpha_n = \alpha_0 + n\lambda$ makes u_n as a power series that can be annihilated with a known annihilating filter given by its z- transform

$$H(z) = \sum_{k=0}^{M} h_k z^{-k} = \prod_{m=0}^{M} (1 - e^{i\lambda m} z^{-1}).$$

Hence, the problem reduces to finding the polynomial coefficients (31), from the annihilation system given by $\{h * u\}_n = 0$ so that the zeros of the polynomial will give the locations $\xi_m = x_m + iy_m$ for $m = 1, \ldots, M$ provided that ξ_m's are distinct. □

It is worth mentioning that the choice of the projection plane that is determined by the holomorphic function is arbitrary and it could have been chosen as XZ or YZ planes rather than XY plane. Indeed, any orientation can be achieved by the sensing functions using standard rotation matrices about x-, y-, and z-axis which will be further developed in the following section.

3 Proof-of-concept validation

In this chapter, we develop a practical algorithm to retrieve the parameters of a stream of 3-D Dirac distribution from the samples of induced field on a given measurement boundary. In particular, we choose to work with a specific holomorphic function that introduces a first-order pole at the origin; i.e., using the convention from the previous section we choose $\varphi(\xi) = \xi$ where the complex variable is defined as $\xi = x + iy$. Moreover, we provide the details of the implementation with experimental results.

3.1 Sensing Step

In Section 2 we proposed to use novel sensing kernels that are derived from holomophic functions in complex domain. Now, we consider a general spherical sampling geometry and restrict our choice of the sensing function based on the following lemma.

Lemma 3.1. *Let $a \in \mathbb{C}$ and $\mathbf{r} = [x, y, z]^T$*

$$\psi(x, y, z) = \frac{e^{i\omega z/c}}{x + iy - a}, \; a \notin \Omega \tag{33}$$

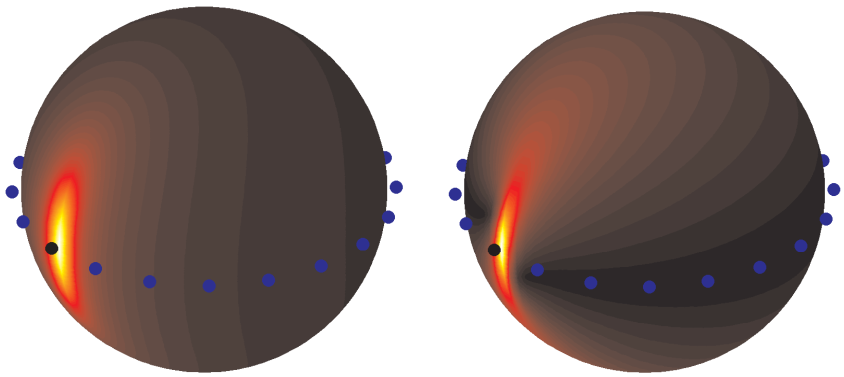
is a valid sensing function that belongs to the space of functions defined by (11)*. The proposed test function is visualized on the measurement surface in Fig. 2.*

3.2 Annihilation Step

Proposition 3.2. *Let ψ_n be a family of sensing functions each as in Lemma 3.1 for $n = 0, \cdots, N-1$ with a_n's located with equidistant radial angle, θ on the complex plane, then, the set of generalized samples can be annihilated to find the projections of the source points onto complex plane.*

Proof. Choosing N points on the complex plane to define the family of sensing functions in the form $a_n = \alpha_n e^{in\theta}$ with α_n's are greater than the radius of the measurement surface to satisfy the Lemma 3.1 and θ is an arbitrary angle.

Then, defining a polynomial $R(a_n) = \sum_{m=0}^{M} r_m a_n^m = \prod_{m=1}^{M}(x_m + iy_m - a_n)$ with $r_M = 1$ and an FIR filter, h with zeros at $e^{ik\theta}$ given by $H(z) =$



(a) (b)

Figure 2: Visualalization for the sensing function on the measurement surface Ω where the dots around the volume indicate the different sensing positions, (i.e., a_n) (a) sensing function (b) normal derivative of the sensing function corresponding to sensing position shown in dark

$\sum_{k\in\mathbb{Z}} h[k]z^{-k} = \prod_{k=0}^{M-1}(1 - e^{ik\theta}z^{-1})$, then given the set of generalized samples can be reinterpreted as

$$\begin{aligned}\langle \psi_n, Q\rangle &= \sum_{m=1}^{M} c_m(\omega)\frac{e^{ikz_m}}{x_m + iy_m - a_n},\\ &= \frac{\sum_{m=0}^{M-1} c'_m e^{inm\theta}}{\prod_{m=1}^{M}(x_m + iy_m - a_n)},\\ &= \frac{\sum_{m=0}^{M-1} c'_m e^{inm\theta}}{R(a_n)}.\end{aligned} \tag{34}$$

Then, the predefined filter h annihilates the sequence, $u_n = \{R(a_n)\mu_n\}$ for $n = M, \dots, N-1$ where $\mu_n \langle\psi_n, Q\rangle$

$$\begin{aligned}0 = \{h * u\}_n &= \sum_{n'=0}^{N-1} h_{n-n'}R(a_{n'})\mu_{n'}\\ &= \sum_{n'=0}^{N-1} h_{n-n'}\sum_{k=0}^{M} r_k a_{n'}^k \mu_{n'}\\ &= \sum_{k=0}^{M} r_k \sum_{n'=0}^{N-1} h_{n-n'} a_{n'}^k \mu_{n'}\\ &= \sum_{k=0}^{M} A_{n,k} r_k.\end{aligned} \tag{35}$$

□

In matrix representation, (35) can be represented as

$$\mathbf{Ar} = \mathbf{HDVr} = \mathbf{0}, \tag{36}$$

where $\mathbf{H}$ is an $(N-M)\times N$ Toeplitz matrix representing the annihilating filter h, $\mathbf{D}$ is an $N\times N$ diagonal matrix of the generalized samples, $\mathbf{V}$ is an $N\times(M+1)$ Vandermonde matrix of poles of the sensing function and $\mathbf{r}$ is the unknown vector of $M+1$ polynomial coefficients with $r_M = 1$. Once the unknown polynomial coefficients satisfying (36) are obtained, the projection of the source point onto complex plane are found as the roots of the polynomial $R(a_n)$.

Lemma 3.3. *Let the locations of the point sources be distinct, then the system matrix in* (36) *is of rank M for the noiseless case.*

Proof. The system matrix in (36) has $(N-M)$ equations with M unknowns of the characteristic polynomial $R(a_n)$ defined in Proposition 3.2 with $r_M = 1$. Then, the minimum number of generalized samples should be $N = 2M$. Hence, we conclude that the system matrix $\mathbf{A}$ in (36) is rank M for distinct source position. □

3.3 Practical Recovery in 3-D

We propose a three-step algorithm to locate the 3-D locations of the point sources from the measured field by means of applying the sensing principle.

3.3.1 Planar Projection

In the first step, we choose a set of sensing functions Ψ as in (33) in a general X'Y'Z' coordinate system that we represent with general rotation matrices along X and Y-axes [3].

$$\Psi_n(\mathbf{Rr},\omega) = \frac{e^{j\omega z'/c}}{x' + jy' - a_n},\ a_n \notin \Omega, \tag{37}$$

where a_n's are the poles of the sensing function on X'Y'-plane located at equidistant angles $a_n = ae^{jn\theta}, n \in [\![0, N-1]\!]$, $|a|$ is greater than the radius of Ω excluding the volume and θ is an arbitrary angle. The matrix $\mathbf{R}$ represents rotation matrix of the coordinate system along the X and Y axes in a standard right-handed cartesian coordinate system given by

$$\underbrace{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}}_{\mathbf{r}'} = \underbrace{\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}}^{\mathbf{R_X}(\alpha)} \overbrace{\begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}}^{\mathbf{R_Y}(\beta)}}_{\mathbf{R}(\alpha,\beta)} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{r}}. \tag{38}$$

Then, solving the annihilation system in Proposition 3.2, we find the projections of the point absorbers' positions on the corresponding X' Y'-plane defined by the rotation matrix.

3.3.2 Pairing of the Projections

In the second step, we propose a greedy approach to build a manifold of projections to keep the solutions paired between each projection. In particular, the 2-D projections of the locations are the roots of a polynomial as described in Proposition 3.2. Thus, the projections are not properly ordered. Indeed, the problem gets even more significant for multiple point source recovery and an efficient way to handle the problem becomes critical. In the ideal case, at least 2 orthogonal projections of the distribution would be sufficient to solve for the 3-D problem. However, to the best of our knowledge there is no simple solution for this problem under the effect of noise or measurement error.

We propose a solution for the closest pair problem for two separated sets of points between consecutive projection planes. Indeed, the main idea is to compute the Euclidean distance in $\mathbb{R}^3$ between all the pairs of points in two projection sets and then group the pairs with respect to the mutually smallest distance criteria.

Consider $A \times B$ projection planes defined by $R(\alpha_i, \beta_j)$ $i \in [\![0, A-1]\!]$ $j \in [\![0, B-1]\!]$ where each plane has M projected points to be paired. We assume an initial labelling for the points of the first plane with l_1 to l_M. Then, to find the closest pair of points $p \in P_k$ and $q \in P_{k-1}$ $k \in [\![2, A \times B]\!]$, we compute the distances between all the $M \times M$ pairs of points and we pick and label the pair with the smallest distance and exclude it from the set.

In a similar way, the idea can be generalized for a rotation along x and y axis using a selective projection approach. Indeed, we propose to selectively project onto planes such that the incremental change between the planes remains the minimum. Hence, the Euclidean distance still achieves a good measure to pair the projections between two consecutive projections.

We note that in practice, projection angles along x and y- axis do not have to be different due to the fact that the projection is done on a complex plane and only the distance of each source point to the measurement surface matters in reconstruction quality. Hence, choosing the rotation matrix $R(\alpha, \alpha)$ achieves sufficiently good results.

We finally provide a summary of the method in Algorithm 1 and we note that the method is computed in $O(n^2)$ but can be solved in $O(n \log n)$ using the recursive divide and conquer approach [21].

Algorithm 1: Closest Pair of Points

Data: $p \in P_i$, for $i \in [[0, P-1]]$
Result: l_p: Labels of $p \in P_i$
begin
  Initialize: Label l_0: 1 to M
  for *i=1* **to** *P-1* **do**
  $P_i^* = P_i$
  while P_i^* *is not empty* **do**
    $p^* = \underset{p \in P_i^*}{\operatorname{argmin}} \min_{q \in P_{i-1}} ||f(p) - f(q)||^2$
    $P_i^* = P_i^* \backslash \{p^*\}$
  Label l_i : Match the labels of p^* and q
  ;

3.3.3 Reconstruction in 3-D

In the third step, we solve for the 3D positions of the point absorbers by a least-squares regression of the 2D projections as a special case of tomographic reconstruction. Indeed, once the pairing of the projections for each rotation matrix is known after the second step of the algorithm, one can represent each 2D projections with the

$$\underbrace{\begin{bmatrix} x_m^k \\ y_m^k \end{bmatrix}}_{\xi_m^k} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{R}(\alpha_i, \beta_j)}_{\mathbf{P}^k} \underbrace{\begin{bmatrix} x_m \\ y_m \\ z_m \end{bmatrix}}_{\mathbf{r}_m}, \tag{39}$$
$$k = i \times A + j,\ i \in [\![0, A-1]\!],\ j \in [\![0, B-1]\!]$$

where $\mathbf{R}(\alpha_i, \beta_j)$ characterize a set of rotations, k is the index for the selective projection order and ξ_m^k is the projection of the point absorber $\mathbf{r}_m$ on the plane denoted by k. Finally, we solve for the following least squares problem

$$\hat{\mathbf{r}}_m = \underset{\mathbf{r}_m}{\operatorname{argmin}} \sum_{i=0}^{P-1} \|\xi_m - \mathbf{P}^k \mathbf{r}_m\|_2^2,\ \forall \text{m} \in [\![1, M]\!]. \tag{40}$$

3.3.4 A note on the missing Fourier coefficients

In order to completely describe the source distribution, one still has to determine the temporal Fourier coefficients $s_m(\omega)$. The estimation of these parameters can be done with any set of *generalized samples*. Considering the estimated locations, the generalised samples will be a linear set of equations to be solved for $s_m(\omega)$

$$\mu_n = \langle \psi_n, Q \rangle = \sum_{m=1}^{M} s_m(\omega) \underbrace{\frac{e^{ikz_m}}{x_m + iy_m - a_n}}_{estimated}, \; n \in [\![1, N]\!], \tag{41}$$

where μ_n are the generalized samples and x_m, y_m and z_m are the estimated 3-D positions of the source.

Overview of the Algorithm

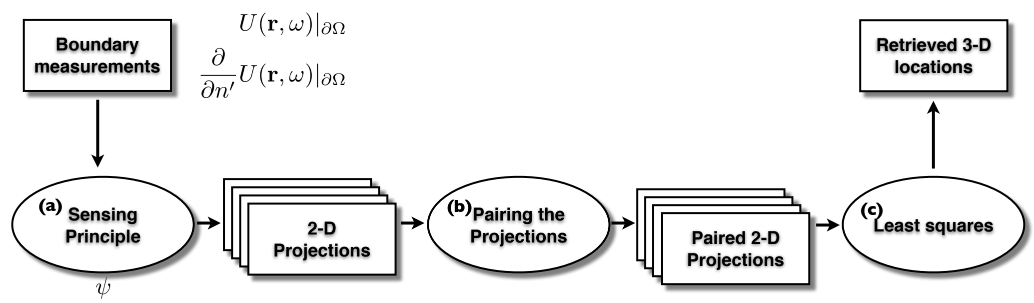


Figure 3: A schematic of the proposed algorithm to retrieve the locations of a 3-D stream of Diracs from the boundary measurements of an induced field: (a) Find the projected positions on several complex planes by the sensing principle; (b) Pair the projections between each projection plane with respect to the Euclidean distance; (c) Retrieve the 3-D locations by solving a least-squares regression problem.

In this part, we recapitulate the proposed algorithm for the recovery of parameters of a source distribution given as a set of 3-D Diracs. In Figure 3, a flow-chart of the proposed algorithm is provided.

3.4 Experimental Results

We performed numerical experiments to validate our algorithm. Specifically, we considered a spherical geometry of the measurement surface that is assumed to enclose the source function $Q(\mathbf{r}, \omega) = \sum_{m=1}^{M} s_m(\omega)\delta(\mathbf{r} - \mathbf{r_m})$ to be determined. Here, we note that there has been extensive research [6, 17, 19, 20] on the problem that deals with the analysis and design of spherical sensor arrays to allow aliasing-free spatial sampling of the data, if possible, or otherwise with minimal spatial aliasing. However, we would like to repeat (12) here to emphasize the fact that the sensing principle relies on the generalized samples of the source function by the following surface integral that links the sampled measurements to the model parameters

$$\mu_n = \langle \psi_n, Q \rangle = \oint_{\partial\Omega} \left[\psi_n(\mathbf{r}, \omega) \frac{\partial}{\partial n} U(\mathbf{r}, \omega) - U(\mathbf{r}, \omega) \frac{\partial}{\partial n} \psi_n(\mathbf{r}, \omega) \right] dS \qquad (42)$$

where the partial derivatives $\frac{\partial}{\partial n}$ are directed outward from the interior to exterior. For the current work, we consider that the integral in (42) can be taken numerically and we will provide a more advanced method of the approximation of (42) in future work that can also accommodate missing data.

Once the generalised samples have been acquired with the sensing step, the annihilation step follows by constructing the linear system of equations in (36) so that the projections on a plane can be retrieved. As we have seen in Section 3.2, we first use the lemma 3.3 to determine the model order M. In Fig. 4 (a), we demonstrate that it is possible to estimate the model order M by observing the decomposition of singular values of the system matrix $\mathbf{A}$ in (36). For a 3-D retrieval of the projections, we follow the instruction as in Section 3.3. And in Figure 4, we provide a case where the true model has five point sources.

4 Discussion and Conclusion

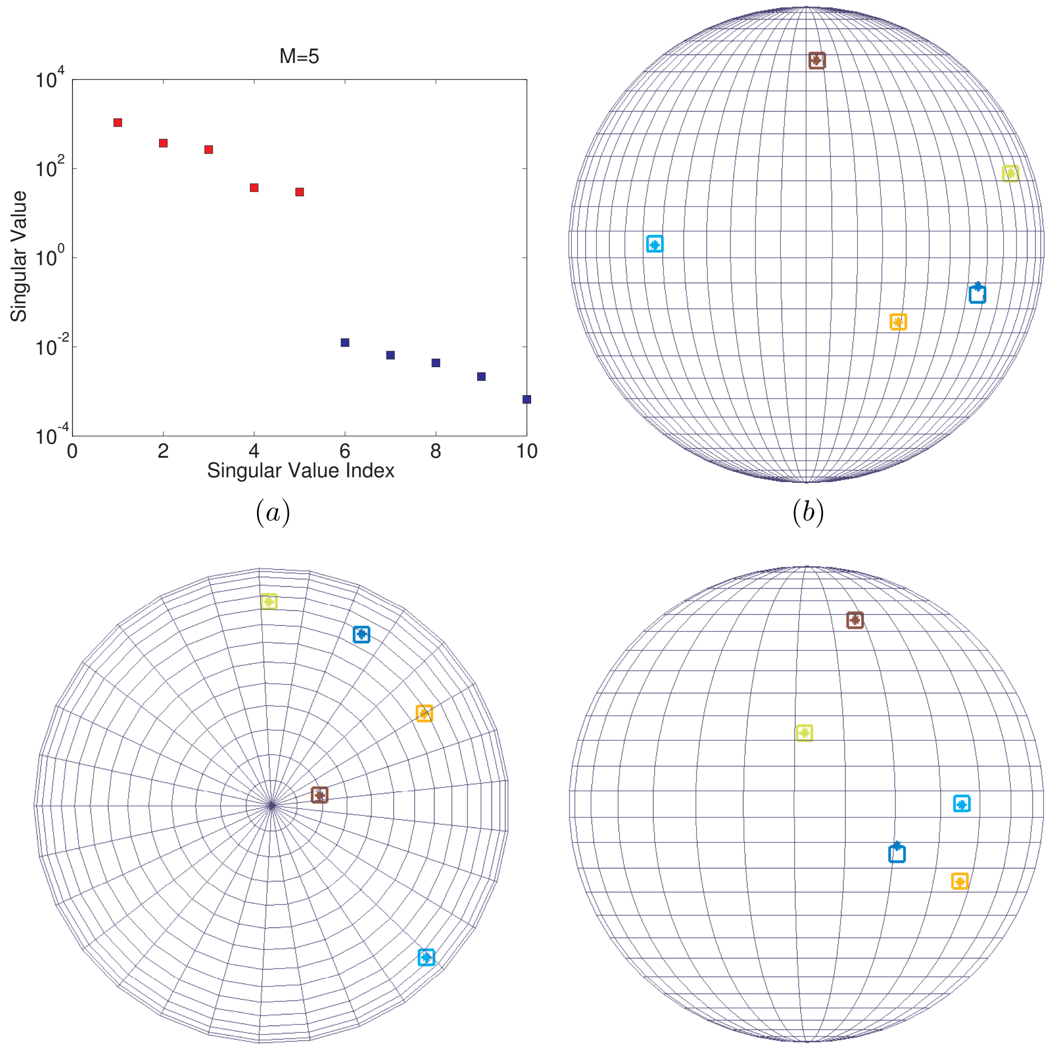
To summarize, we proposed a novel FRI-like algorithmic framework for identifying parametric source models from boundary measurements of a radiating field. We proposed to use novel sensing functions that are derived from holomorphic functions which allow to split the 3–D localization problem into several 2–D projections onto planes defined by the holomorphic function.

Introducing the zeros of the holomorphic function that generates an N-th degree complex polynomial as the poles of the sensing function, we achieved a locally selective sensing function that is capable to spatially select the influence of the nearby point sources. This property is important in practice since the full view of the field data is usually not available in real applications. Therefore, this enables us to have better approximation of the closed surface integral of the measurements that is the fundamental equation of the sensing principal.

We demonstrated the feasibility of the proposed algorithm by experimental results and our future research will focus on real applications.

5 Acknowledgments

This work was supported in part by the Swiss National Science Foundation (under the grants 205330-132808 and PP00P2-146318) and in part by the Center for Biomedical Imaging (CIBM) of the Geneva-Lausanne Universities and the EPFL and in part by the Hong Kong University Grant Council (under the RGC grant CUHK410110).



(a) (b)

(c) (d)

Figure 4: Point Source Model Estimation: (a) Estimation of model order by separation of singular values, Different Planar views of the true (+) and the estimated (□) locations of the source in (b) YZ-plane view (c) XY-plane view (d)XZ-plane view where the colors are showing the estimated and true magnitude of the corresponding point source.

References

[1] *Introductory Functional Analysis with Applications.* Wiley classics library. Wiley India Pvt. Limited, 2007.

[2] Stphane Andrieux and Amel Ben Abda. Identification of planar cracks by complete overdetermined data: inversion formulae. *Inverse Problems*, 12(5):553, 1996.

[3] G.G.B. Arfken and H.J. Weber. *Mathematical Methods for Physicists*. Academic Press, 2005.

[4] Abdellatif El Badia and Takaaki Nara. An inverse source problem for helmholtz's equation from the cauchy data with a single wave number. *Inverse Problems*, 27(10):105001, 2011.

[5] Richard Baraniuk. Compressive radar imaging. In *Proc. 2007 IEEE Radar Conf*, pages 128–133, 2007.

[6] I. Ben Hagai, F.M. Fazi, and B. Rafaely. Generalized sampling expansion for functions on the sphere. *Signal Processing, IEEE Transactions on*, 60(11):5870–5879, 2012.

[7] T. Blu, P.-L. Dragotti, M. Vetterli, P. Marziliano, and L. Coulot. Sparse sampling of signal innovations. *Signal Processing Magazine, IEEE*, 25(2):31 –40, march 2008.

[8] Ingrid Daubechies. *Ten lectures on wavelets*. Society for Industrial and Applied Mathematics, 1 edition, June 1992.

[9] A.J. Devaney. *Mathematical Foundations of Imaging, Tomography and Wavefield Inversion*. Mathematical Foundations of Imaging, Tomography and Wavefield Inversion. Cambridge University Press, 2012.

[10] Z. Dogan, I. Jovanovic, T. Blu, and D. Van De Ville. Localization of point sources in wave fields from boundary measurements using new sensing principle. In *Proceedings of the Tenth International Workshop on Sampling Theory and Applications (SampTA'13)*, July 1–5, 2013.

[11] P.-L. Dragotti, M. Vetterli, and T. Blu. Sampling moments and reconstructing signals of finite rate of innovation: Shannon meets strang-fix. *Signal Processing, IEEE Transactions on*, 55(5):1741–1757, 2007.

[12] Albert Fannjiang, Thomas Strohmer, and Pengchong Yan. Compressed remote sensing of sparse objects. *SIAM J. Imaging Sciences*, 3(3):595–618, 2010.

[13] D. Finch and S. Patch. Determining a function from its mean values over a family of spheres. *SIAM Journal on Mathematical Analysis*, 35(5):1213–1240, 2004.

[14] V. Isakov. *Inverse Problems for Partial Differential Equations*. Number v. 127 in Applied Mathematical Sciences (Springer-Verlag), Vol 127. Springer, 1998.

[15] D. Kandaswamy, T. Blu, and D. Van De Ville. Analytic sensing: Noniterative retrieval of point sources from boundary measurements. *SIAM J. Sci. Comput.*, 31(4):3179–3194, July 2009.

[16] Djano Kandaswamy. *Sparse Source Recovery from Boundary Measurements using an Extension of Prony's Method for the Poisson Equation*. PhD thesis, STI, Lausanne, 2011.

[17] J.D. McEwen and Y. Wiaux. A novel sampling theorem on the sphere. *Signal Processing, IEEE Transactions on*, 59(12):5876–5887, 2011.

[18] Jon Oativia, Simon R Schultz, and Pier Luigi Dragotti. A finite rate of innovation algorithm for fast and accurate spike detection from two-photon calcium imaging. *Journal of Neural Engineering*, 10(4):046017, 2013.

[19] B. Rafaely. Analysis and design of spherical microphone arrays. *Speech and Audio Processing, IEEE Transactions on*, 13(1):135–143, 2005.

[20] B. Rafaely, B. Weiss, and E. Bachmat. Spatial aliasing in spherical microphone arrays. *Signal Processing, IEEE Transactions on*, 55(3):1003–1010, 2007.

[21] Michael Ian Shamos and Dan Hoey. Closest-point problems. In *Foundations of Computer Science, 1975., 16th Annual Symposium on*, pages 151 –162, oct. 1975.

[22] Pancham Shukla and Pier Luigi Dragotti. Sampling schemes for multidimensional signals with finite rate of innovation. *Signal Processing, IEEE Transactions on*, 2006.

[23] P. Stoica and R.L. Moses. *Introduction to spectral analysis*. Prentice Hall, 1997.

[24] R. Tur, Y.C. Eldar, and Z. Friedman. Innovation rate sampling of pulse streams with application to ultrasound imaging. *Signal Processing, IEEE Transactions on*, 59(4):1827–1842, April 2011.

[25] J.A. Uriguen, T. Blu, and P.L. Dragotti. Fri sampling with arbitrary kernels. *Signal Processing, IEEE Transactions on*, 61(21):5310–5323, 2013.

[26] D. Van De Ville, T. Blu, B. Forster, and M. Unser. Isotropic-polyharmonic b-splines and wavelets. In *Image Processing, 2004. ICIP '04. 2004 International Conference on*, volume 1, pages 661–664 Vol. 1, 2004.

[27] M. Vetterli, P. Marziliano, and T. Blu. Sampling signals with finite rate of innovation. *Signal Processing, IEEE Transactions on*, 50(6):1417 –1428, june 2002.

[28] ZongMin Wu and JianPing Liu. Generalized strang-fix condition for scattered data quasi-interpolation. *Advances in Computational Mathematics*, 23(1-2):201–214, 2005.