On the N-Dimensional Extension of the Discrete Prolate Spheroidal Window

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***Abstract*—The optimal one-dimensional (1-D) window, an index-limited sequence with maximum energy concentration in a finite frequency interval, is related to a particular discrete prolate spheroidal sequence. This letter presents the N-dimensional (N-D) extension, i.e., the N-D window with limited support and maximum energy concentration in a general nonseparable N-D passband. These windows can be applied to multidimensional filter design and the design of optimal convolution functions. We show the three-dimensional (3-D) optimal window based on a rhombic dodecahedron as a passband region.**

***Index Terms*—Discrete prolate spheroidal sequences, multidimensional signal processing.**

I. Introduction

THE OPTIMAL window is an index-limited sequence with maximum energy concentration in a finite frequency interval. Slepian [1] showed that this window is related to the zeroth discrete prolate spheroidal sequence (DPSS), which we refer to as the discrete prolate spheroidal window (DPSW). The computation of the one-dimensional (1-D) DPSW is well-known [2]–[4] and the window has many applications in signal and image processing (such as filter design [5], spectrum analysis [6], [7], extrapolation [8], gridding algorithms for MRI [9], SAR imaging).

First, we briefly review the standard method to compute the 1-D DPSW in Section II. Next, we generalize this method to N dimensions in Section III. The multidimensional extension described by Slepian [10] only treats the spherical (continuous) solution, while our approach can be used for a general nonseparable N-D passband. Since the order of the related eigenvalue problem increases very fast for higher dimensions, we show how to reduce the order by 2^N. Next, we present a three-dimensional (3-D) DPSW based on a rhombic dodecahedron in Section IV.

II. One-Dimensional Digital Prolate Spheroidal Window

Consider a real-valued index-limited sequence h_n, $n = 0, 1, \ldots, N_1 - 1$; we can compute its frequency response using Fourier series: $H(f) = \sum_{n=0}^{N_1-1} h_n e^{-j2\pi fn}$. We wish to obtain those h_n, such that the energy proportion in a given passband $[-W, W]$, $W \in [0, (1/2))$, is maximized

$$\lambda = \frac{\int_{-W}^{W} |H(f)|^2 \, df}{\int_{-\frac{1}{2}}^{\frac{1}{2}} |H(f)|^2 \, df}. \tag{1}$$

Using Parseval's theorem we have that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |H(f)|^2 \, df = \sum_{n=0}^{N-1} |h_n|^2 \tag{2}$$

which implies that (1) can be expressed as a ratio of quadratic forms; in matrix notation

$$\lambda = \frac{\mathbf{h}^T \mathbf{A} \mathbf{h}}{\mathbf{h}^T \mathbf{h}} \tag{3}$$

where $\mathbf{h} = [h_0 \; h_1 \; \cdots \; h_{N_1-1}]^T$ and $\mathbf{A}$ is a real and symmetric kernel

$$[\mathbf{A}]_{m,n} = \frac{\sin(2\pi W(m-n))}{\pi(m-n)}. \tag{4}$$

The DPSS $\mathbf{h}^{(k)}(N_1, W)$ are the normalized eigenvectors of the kernel matrix $\mathbf{A}$

$$\sum_{n=0}^{N_1-1} \frac{\sin(2\pi W(m-n))}{\pi(m-n)} h_n^{(k)} = \lambda^{(k)} h_m^{(k)}. \tag{5}$$

This system has N_1 distinct eigenvalues and eigenvectors which are normalized such that

$$\sum_{n=0}^{N_1-1} \left(h_n^{(k)}\right)^2 = 1, \quad \text{and} \quad \sum_{n=0}^{N_1-1} h_n^{(k)} \geq 0. \tag{6}$$

The DPSW is the eigenvector $\mathbf{h}^{(k)}$ which corresponds to the largest eigenvalue $\lambda^{(k)}$, and thus has the best energy concentration in the given frequency interval.

This procedure can also be used to approximate the (continuous) prolate spheroidal wave functions (PSWF) by increasing N_1 while keeping the product $N_1 W$ constant [1].

III. N-Dimensional Digital Prolate Spheroidal Window

We now extend the basic approach of Section II to N dimensions. In that case, we consider an N-D index-limited sequence $h_{\mathbf{n}}$ with $\mathbf{n} \in \mathcal{A} = \{(n_1, n_2, \ldots, n_N) \,|\, n_i = 0, 1, \ldots, N_i - 1\}$. The frequency response of $h_{\mathbf{n}}$ can be written as $H(\mathbf{f}) = \sum_{\mathbf{n} \in \mathcal{A}} h_{\mathbf{n}} e^{-j2\pi \langle \mathbf{f}, \mathbf{n} \rangle}$ using the inner product $\langle \mathbf{f}, \mathbf{n} \rangle = \sum_{i=1}^{N} f_i n_i$. The window which has maximal energy

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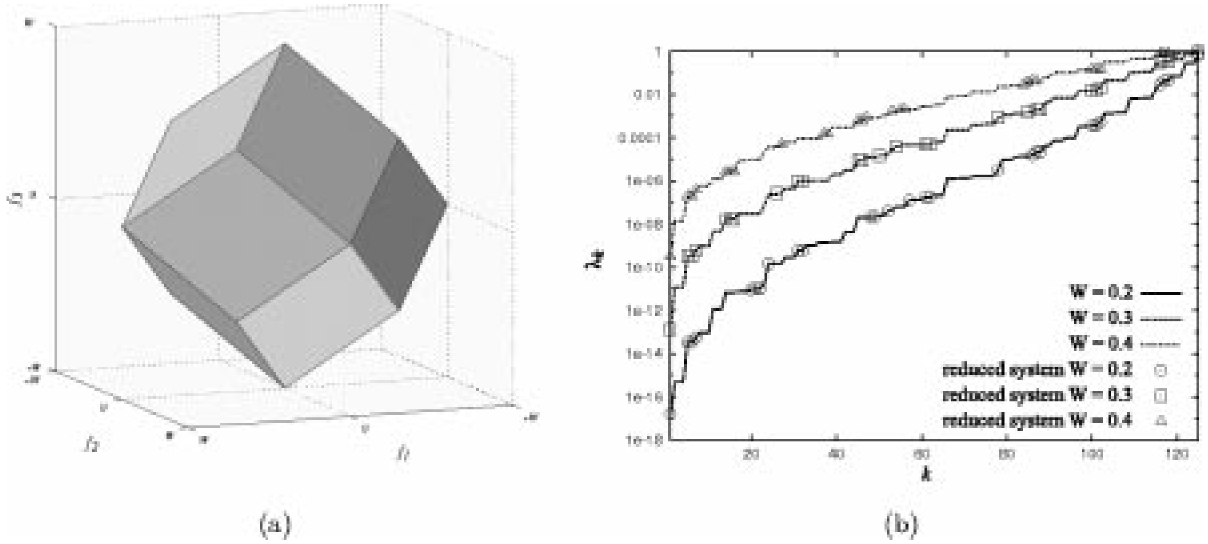

Fig. 1. (a) Rhombic dodecahedron that is used as a 3-D passband region and (b) the eigenvalues of the 3-D DPSS problem for $N_1 = N_2 = N_3 = 5$ and $W = 0.2, 0.3, 0.4$ using the dodecahedron passband region in the normal and reduced system.

concentration in a general N-dimensional (N-D) passband region $\mathbf{W}$ maximizes the ratio

$$\lambda = \frac{\int_{\mathbf{W}} |H(\mathbf{f})|^2 \, d\mathbf{f}}{\int_{\mathbf{R}} |H(\mathbf{f})|^2 \, d\mathbf{f}} \tag{7}$$

where $\mathbf{R}$ is the N-D hypercube

$$\mathbf{R} = \left\{ (f_1, f_2, \ldots, f_N) \,|\, -\frac{1}{2} \leq f_i \leq \frac{1}{2} \right\}. \tag{8}$$

Analogously to the 1-D case, Parseval's theorem can be used to obtain

$$\sum_{\mathbf{n} \in \mathcal{A}} K(\mathbf{k} - \mathbf{n}) h_{\mathbf{n}} = \lambda h_{\mathbf{k}}, \quad \forall \mathbf{k} \in \mathcal{A} \tag{9}$$

where $K(\mathbf{u})$ is a N-D kernel, i.e., the inverse Fourier transform of the passband region $\mathbf{W}$

$$K(\mathbf{u}) = \int_{\mathbf{W}} e^{j2\pi \langle \mathbf{u}, \mathbf{f} \rangle} \, d\mathbf{f}. \tag{10}$$

Now, (9) can be rewritten in the standard eigenvalue form by enumerating all coefficients lexicographically. The index i, $i = 0, 1, \ldots, (\prod_{j=1}^{N} N_j - 1) = (L - 1)$ addresses every vector $\mathbf{n} \in \mathcal{A}$, such that

$$\sum_{i=0}^{L-1} K(\mathbf{n}_k - \mathbf{n}_i) h_{\mathbf{n}_i} = \lambda h_{\mathbf{n}_k}, \quad 0 \leq k \leq L - 1. \tag{11}$$

For most applications, the passband region $\mathbf{W}$ is symmetrical around its coordinate axes, and the eigenvector with the largest eigenvalue, i.e., the N-D DPSW, is also symmetric around N symmetry-axes. Therefore, we can reduce the order of the system significantly. Assume all N_i are odd. In the 1-D case, h_n is symmetric around the central window position $n = (N_1 - 1)/2$. The 2-D window h_{n_1,n_2} is symmetrical around the two central axes $n_1 = (N_1 - 1)/2$ and $n_2 = (N_2 - 1)/2$, such that one quadrant determines the whole window. In general, the size of the N-D window can be reduced from $\prod_{i=1}^{N} N_i$ to $\prod_{i=1}^{N}((N_i - 1)/2 + 1)$. In order to incorporate this reduction into the eigenvalue problem, we derive a transformation matrix $\mathbf{T}_N$, which maps a reduced window $\mathbf{w}$ onto the original window $\mathbf{h} = \mathbf{T}_N \mathbf{w}$. The matrix $\mathbf{T}_N$ can easily be constructed recursively

$$\mathbf{T}_0 = [1]$$
$$\mathbf{T}_n = \begin{bmatrix} \mathbf{T}_{n-1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbf{T}_{n-1} \\ 0 & \mathbf{T}_{n-1} & 0 \\ \mathbf{T}_{n-1} & 0 & 0 \end{bmatrix} \tag{12}$$

where $\mathbf{T}_n$ places $((N_n - 1)/2) + 1$ copies of $\mathbf{T}_{n-1}$ on a diagonal, and $(N_n - 1)/2$ copies on the second diagonal below. A proper normalization of the matrix $\mathbf{T}_N$ makes $\mathbf{h}^T \mathbf{h} = \mathbf{w}^T \mathbf{T}_N^T \mathbf{T}_N \mathbf{w}$, preserving the eigenvalues. The normalized matrix $\mathbf{T}'_N$ equals

$$\mathbf{T}'_N = \frac{\mathbf{T}_N}{\left(\mathbf{T}_N^T \mathbf{T}_N\right)^{\frac{1}{2}}} \tag{13}$$

where $\mathbf{T}_N^T \mathbf{T}_N$ is a square diagonal matrix. In the case N_i is even, the middle row of $\mathbf{T}_i$ needs to be omitted.

The eigenvector $\mathbf{w}$ corresponding to the largest eigenvalue of the reduced matrix $\mathbf{B} = \mathbf{T}'^T_N \mathbf{A} \mathbf{T}'_N$ can be computed and converted to $\mathbf{h} = \mathbf{T}'_N \mathbf{w}$. If all N_i are equal to M, the size of the N-D system is reduced by a factor

$$\left(\frac{M}{(M-1)/2 + 1} \right)^N \approx 2^N. \tag{14}$$

IV. Results

We compute the 3-D DPSW corresponding to a rhombic dodecahedron passband region. This dodecahedron is related to a "body-centered cubic" sampling lattice, which has a better sampling efficiency for isotropic processes compared to cubic sampling [11], [12].

TABLE I
RESULTS OF THE COMPUTATION OF THE 3-D DPSW

	order		computation
M	original	reduced	time
5	125	27	< 1s
7	343	64	< 1s
9	729	125	5s
11	1331	216	31s
13	2197	343	140s
15	3375	512	580s

To obtain the kernel function, we integrate the Fourier kernel throughout a regular rhombic dodecahedron of width $\sqrt{2}W/2$ between parallel faces [see Fig. 1(a)]

$$\begin{aligned} K(\mathbf{u}) &= K(u_1, u_2, u_3) \\ &= \frac{1}{\pi^3 \left(u_1^4 + u_2^4 + u_3^4 - 2\left(u_1^2 u_2^2 + u_1^2 u_3^2 + u_2^2 u_3^2\right)\right)} \\ &\quad \times \{-2u_1 \sin(2\pi W u_1) - 2u_2 \sin(2\pi W u_2) \\ &\quad - 2u_3 \sin(2\pi W u_3) \\ &\quad + (u_1 + u_2 + u_3) \sin(\pi W(u_1 + u_2 + u_3)) \\ &\quad + (u_1 - u_2 + u_3) \sin(\pi W(u_1 - u_2 + u_3)) \\ &\quad + (u_1 - u_2 - u_3) \sin(\pi W(u_1 - u_2 - u_3)) \\ &\quad + (u_1 + u_2 - u_3) \sin(\pi W(u_1 + u_2 - u_3))\}. \qquad (15) \end{aligned}$$

To illustrate the proposed approach, Fig. 1(b) plots the eigenvalues of the 3-D DPSS problem for $M = N_1 = N_2 = N_3 = 5$ and several values of W. The order of the original system is 125. The reduced system contains those solutions which fullfil the symmetry requirements of (12), reducing the order to 27. The energy concentration for $W = 0.2, 0.3, 0.4$ is respectively 0.700, 0.945, and 0.994. Table I shows the computation time for several sizes. These measurements where obtained using Matlab on an Intel Pentium II Xeon 450 MHz for reducing and solving the eigensystem. Note that the reduction enables us to compute the $M = 13$ system in approximately the same time as the nonreduced $M = 7$ system.

V. CONCLUSION

This paper presented an N-D extension for the digital prolate spheroidal window which makes it possible to specify a nonseparable N-D passband. This extension is based on the standard method for the 1-D case. The order of the related eigenvalue problem can be significantly reduced taking into account the symmetry of the DPSW. A proper normalization preserves the eigenvalues so they still represent the energy ratio. The paper also includes as an example the design of a 3-D DPSW based on a rhombic dodecahedron passband region.

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